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Interpreting Quantum Mechanics according to a Pragmatist Approach

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Abstract The aim of this paper is to show that quantum mechanics can be interpreted according to a pragmatist approach. The latter consists, first, in giving a pragmatic definition to each term used in microphysics, second, in making explicit the functions any theory must fulfil so as to ensure the success of the research activity in microphysics, and third, in showing that quantum mechanics is the only theory which fulfils exactly these functions.

Keywords Quantum mechanics · Interpretation · Pragmatism · Instrumentalism

1 Introduction

Many interpretations of quantum mechanics (QM) aim to answer the question: "How is the world according to QM?" (see e.g. Healey [1] and van Fraassen [2]). The most known of them are the pilot-wave theory [3, 4], the spontaneous reduction theory [5, 6], the many-worlds interpretations [7–10], the many-minds interpretations [11, 12], the modal interpretations [1, 2, 13, 14], and the consistent histories interpretations [15–17]. Although they seem to solve the so-called "measurement problem",[1] these interpretations have many disadvantages: (a) they modify the formalism of QM (e.g. the pilot-wave theory postulates a second equation of evolution or the spontaneous reduction theory modifies Schrödinger's equation) without leading to new

[1] For a discussion of this problem, see e.g. van Fraassen [2, Chap. 9], d'Espagnat [18], Mittelstaedt [19], and Bächtold [20].

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testable predictions; (b) they face new theoretical problems (e.g. many of them conflict with special relativity theory); and (c) they sometimes offer "extravagant" images of the world (e.g. the "universe" as consisting of an infinite number of parallel "worlds").

Furthermore, there is no conclusive argument in favour of one of the multiple images of the world proposed by these interpretations. All of them are equally defensible, or rather equally exposed to critics. Yet, this *multiplicity* doesn't seem to bring a better understanding of QM, contrarily to what van Fraassen has claimed [2, pp. 481–482]. For, the different images of the world rest on contradictory ideas. The pilot-wave theory and the many-worlds interpretation, for instance, assume that the evolution of the universe is completely deterministic, while there are some intrinsically random processes according to the spontaneous reduction theory. The many-worlds interpretation assumes that every process is local, whereas the pilot-wave theory implies the existence of non-local actions. And so on. How could antagonist ideas contribute to improve our understanding of the theory?

All this calls into question the very possibility of deriving a representation of the world from QM. Instead of saying that this theory *underdetermines* its interpretation, and hence, the representation of the world which can be derived from it (see van Fraassen [2, p. 481], shouldn't we admit that it *does not determine* a representation of the world at all? If this is right, what does it mean to "interpret" QM? The answer I will support in this paper is the following: to "interpret" QM means to *bring to light the usefulness of each term and mathematical component of this theory for the practice of the physicists in microphysics*—the "terms" being used in QM are for instance "preparation", "measurement", "observable", or "microscopic system", while the "mathematical components" of QM are the vector spaces, the complex numbers, Born rule, Schrödinger's equation, the projection postulate, etc. In order to understand why QM fits so well with the experiments, there is no need to appeal to an "external world" with which the theory is allegedly in correspondence. My claim is that QM can be understood in every detail merely by resituating it within the context of the physicists' practice. The argument is simple: it is within the context of the physicists' practice that QM has been elaborated. By rejecting any metaphysical interpretation and paying attention to the relation between the theory and the practice, I am endorsing a "pragmatist" approach in the line of Charles Sanders Peirce [21] and William James [22].

The aim of this paper is to show more concretely how such a *pragmatist interpretation of QM* can be carried through. We will proceed in three steps which consist:

- first, in making explicit the pragmatic meaning of the terms used by the researchers in microphysics, and giving them a "pragmatic definition",
- second, in determining the functions any theory must fulfil so as to ensure the success of the research activity in microphysics (these functions will be called the "pragmatic functions"),
- and third, in showing that QM is the only theory which fulfils exactly the pragmatic functions, and this, (i) by explaining why each mathematical component of QM is needed in order to fulfil the pragmatic functions, and (ii) by verifying that any modification of QM implies either that the pragmatic functions are not all fulfilled, or that the theory entails a useless formal structure.

For this purpose, we will take advantage of the works of several specialists of QM, like R.I.G. Hughes [23], Bas van Fraassen [2], Asher Peres [24], Michel Bitbol [25] and Leslie Ballentine [26], who do not exactly support a "pragmatist" interpretation of QM, but whose analyses of the theory nonetheless are of great interest for a pragmatist approach.

The interpretation proposed in this paper has an "instrumentalist" flavor, and for this reason, can be viewed as similar to those proposed by Günter Ludwig [27, 28], David Finkelstein [29], Asher Peres [24], Willem de Muynck [30], or Chris Fuchs [31]. However, all these authors (except Ludwig) are also claiming some metaphysical ideas, and hence, do not support an entirely instrumentalist view.[2] By contrast, I will try to advocate for an interpretation of QM which is thoroughly pragmatist.

2 First Step: The Pragmatic Definitions

To begin with, let us clarify what exactly the expression "pragmatic meaning" means. In the spirit of Peirce's "pragmatist maxim",[3] I propose the following definition: the "pragmatic meaning" of a word is given by the set of practical effects that can be deduced from it, by all the members of a community, when this word is used in a certain context. With the specification "by all the members of a community", the pragmatic meaning is defined as being *intersubjectively shared*. Although it is intersubjectively shared, this pragmatic meaning remains often implicit. This is the case with the words used by the physicists in the context of microphysics.

Now, the first step of the pragmatist interpretation of QM consists precisely in bringing to light the pragmatic meaning of these words, beyond the ontological images they may suggest. For clarity, each of them will be given a "pragmatic definition" (i.e. a definition making explicit its pragmatic meaning).

Let us warn that the definitions proposed in this section are not meant to be definitive. The purpose here is first to show that the pragmatic meaning indeed exists and can in principle be made explicit; and second, to get unambiguous definitions enabling us, in the second step of the interpretation of QM, to give a precise formulation to the pragmatic functions.

[2] For instance, Finkelstein does not simply reject the traditional substance ontology, but supplies a new one, namely an ontology of actions: "nature is composed of elementary actions, possibly with unpredictable consequences, rather than of elementary objects" [29, p. 26]. Peres believes in the existence of correlations, not only between measurement outcomes, but also between microscopic systems [24, p. 116], he believes that the world is non-local (Ibid., pp. 172–173), and he interprets indeterminism in an ontological manner (Ibid., p. 6). De Muynck thinks that "it is hardly possible to doubt the atomic constitution of matter" or the existence of "electrons and most other elementary particles" [30, p. 78]. As for Fuchs, he writes that "the quantum system represents something real and independent of us" [31, p. 989]; and his aim is to grasp "what quantum mechanics is trying to tell us about nature itself" (Ibid., p. 990).

[3] "Consider what effects, that might conceivably have practical bearings, we conceive the object of our conception to have. Then, our conception of these effects is the whole of our conception of the object" [21, vol. V, § 5.402].

2.1 Preparation, Measurement and Intermediate Phase

In order to describe the different steps of an experiment in microphysics, the physicists usually make a distinction between "preparation" and "measurement". This distinction has been introduced by Margenau as follows [32]:

> In general, a preparation of state is any physical operation which assures that, if a system has been subjected to the operation, it will afterward be in a specific quantum state (ψ, or perhaps a mixture of ψ's). A measurement, on the other hand, will be taken to be a physical operation which yields a *number* (within the penumbra of an error range) that refers to the state present before the operation.

These definitions refer to what the physicists are doing in practice when performing a "preparation" and a "measurement". However, in this quotation, Margenau is using the words "state" and "system" without giving them any pragmatic definition. Hence, Margenau's definitions of a "preparation" and a "measurement" are not themselves completely pragmatic.

More recently, Peres has proposed definitions without making use of the words "state" and "system" and referring only to the physicist's practice [24, p. 12]:

> A preparation is an experimental procedure that is completely specified, like a recipe in a good cookbook. [...] Preparation rules should preferably be unambiguous, but they *may* involve stochastic processes, such as thermal fluctuations, provided that the statistical properties of the stochastic process are known, or at least reproducible.
>
> A test [i.e. measurement] starts like a preparation, but it also includes a final step in which *information*, previously unknown, is supplied to an *observer* (i.e., the physicist who is performing the experiment).

According to Peres, to make a preparation consists concretely in performing a *sequence of experimental operations*.

This is true. But one can be more precise concerning the features of a preparation: it has certain *observable consequences*, that is, consequences that can be observed for instance by means of our eyes. Now, the occurrence of these observable consequences depends on the experimental operations which are performed after the preparation. Relatively to all possible experimental operations that can be performed after the preparation, the latter has a whole *set of potential observable consequences*.

A further point concerning preparations has to be stressed: there are generally many different sequences of experimental operations (for instance, in different laboratories, with different instruments...) having exactly the same set of potential observable consequences. Now, when physicists speak of a given "preparation" and maybe describe a given sequence of operations, what matters for them is not this specific sequence of operations, but rather its potential observable consequences. That is to say, they tacitly refer to *all the sequences of operations* leading to the same set of potential observable consequences.

These considerations lead to the following pragmatic definition:

D_1: A "preparation", noted P, refers to a set of sequences of experimental operations having the same set of potential observable consequences.

When they refer to a certain preparation P, the physicists can avoid mentioning the specific sequence of experimental operations they are performing in their laboratories. They can concentrate on the intersubjectivelly acknowledged set of potential observable consequences of these experimental operations.

In case the physicists perform a sequence of experimental operations and know that it instantiates a certain preparation P, we will say by definition: "they have a *complete knowledge* of the preparation they are performing". In case they know that it instantiates one of the preparations $\{P_i\}$ without knowing exactly which one, we will say by definition: "they have an *incomplete knowledge* of the preparation they are performing".

What about the notion of measurement (or "test")? As Peres has noted, a measurement "starts like a preparation", since it corresponds also to a sequence of experimental operations. By contrast with a preparation, however, a measurement ends up with the production of an outcome. In others words, we are no more dealing with a set of potential observable consequences, but with a unique observable consequence which has occurred as a matter of fact.

For a more precise description of an experiment in microphysics, it is important to make the distinction (not made by Peres) between, on the one hand, the "measurement", which includes the sequence of experimental operations and the occurrence of an outcome, and on the other hand, the phase during which the physicists observe the outcome and make a statement of it.

Note that a *measurement outcome* in microphysics is never directly observed (with our sensory organs). It is inferred from a *macroscopic event*, which by contrast *can* be directly observed.[4] A typical example of macroscopic event occurring at the end of a measurement is the move of a pointer on the apparatus in a given position.

As the words needed to refer to a measurement outcome in microphysics have not all been defined yet, we cannot spell out the pragmatic definition of a "measurement" in terms of "measurement outcome". Nevertheless, we can express it in terms of "macroscopic event" (the words needed to refer to a macroscopic event being simply those of everyday language):

D_2: A "measurement", noted M, is a sequence of experimental operations producing a macroscopic event.[5]

Between the sequence of experimental operations corresponding to the preparation and the one corresponding to the measurement, a certain lapse of time can flow during which no experimental operation is performed. We can call this phase the "intermediate phase":

D_3: An "intermediate phase" is the phase between the preparation and the measurement, during which no experimental operation is performed.

In brief, an experiment in microphysics consists of four phases as in Fig. 1.

[4]The adjective "macroscopic" refers to the scale of human beings.

[5]We have defined a measurement merely as a sequence of operations, and not as a *set* of sequences of operations. The reason is that we are speaking of a measurement *in general* and not of the measurement *of an observable*. (The word "observable" will be defined in the next section.)

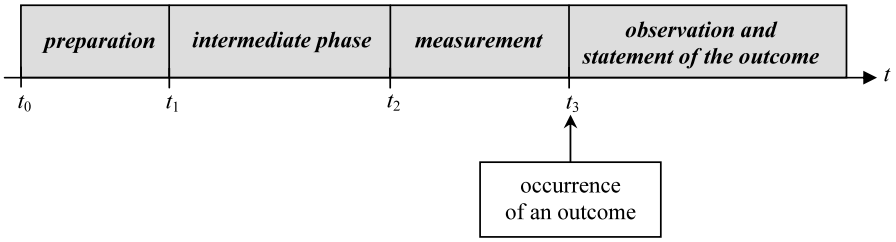

Fig. 1 The four phases of an experiment in microphysics

2.2 Observable

What is measured in microphysics? To answer this question, let us have a look at the potential observable consequences of a measurement. There is an important experimental fact to be put forward: distinct measurements M_α, M_β, $M_\gamma, \ldots$, when producing a certain macroscopic event respectively $\alpha, \beta, \gamma, \ldots$, can be *equivalent* to the same preparation P, i.e. they can lead to the same set of potential observable consequences. Now, this *equivalence* makes it possible to identify a *common object*, usually called an "observable".[6] According to the physicists, this is what a measurement in microphysics is measuring. So, we can give to the term "observable" the following pragmatic definition:

***D*$_4$**: An "observable", noted A, refers to a set of measurements $\{M_\alpha, M_\beta, M_\gamma, \ldots\}$ such that, for all i, the couples (M_α, α_i), (M_β, β_i), $(M_\gamma, \gamma_i), \ldots$, are *equivalent* to the same preparation P_i. Each of these measurements $\{M_\alpha, M_\beta, M_\gamma, \ldots\}$ is said to "measure observable A".

where (M_α, α_i) corresponds to the measurement M_α producing the macroscopic event α_i.

Every macroscopic event occurring at the end of the measurement of an observable enables us to ascribe to this observable a certain *numerical value*,[7] to which a certain *unit* is then associated. The correspondence between the macroscopic event and the numerical value is partly a matter of convention. Nonetheless, once this bi-univocal correspondence is chosen for the spectrum of macroscopic events relative to a given measurement, this bi-univocal correspondence is settled for all the spectra of macroscopic events relative to the measurements of the same observable.

For instance, consider M_α and M_β, both being the measurement of observable A, and such that (M_α, α_i) and (M_β, β_i) are equivalent to the same preparation P_i. If, by convention, α_i is put in correspondence with a certain numerical value of A, say a_i, then, necessary, β_i is also in correspondence with a_i.

[6] One should not confuse this substantive form of the term "observable" with its adjective form.

[7] For purpose of simplicity, we will consider only "sharp" numerical values. However, the discussion can be generalised to the case of "unsharp" or "approximative" numerical values. Concerning experiments providing such unsharp values, see e.g. [24, pp. 417–422], Busch et al. [33], and [30, Sect. 7.10].

2.3 Microscopic System

When describing an experiment in microphysics in terms of "preparation" and measurement", the physicists refer also to a certain "microscopic system", for instance to an "electron" or a "photon". The preparation is assumed to *produce* a certain microscopic system, while the measurement of a given observable is assumed to be *performed on* this system. Consequently, as a first practical effect, the reference to a microscopic system helps the physicists to *conceive the link between a preparation and a measurement* in the context of a certain experiment.

Note that some physicists sometimes speak of the "preparation of a microscopic system in a given *state*" (see for instance the quotation of Margenau given above). We could try to give to this notion of state a pragmatic definition. The problem however is that there doesn't seem to exist an *intersubjective* tacit meaning of this notion. Is the "state" strictly identical to the "state vector"? In this case, we should wait on the discussion about the formalism of QM before introducing the term "state". Or is the "state" referring to the "physical state" of the microscopic system, which determines its properties? In this second case, the problem with regard to the formalism of QM is twofold. First, the way the physical state determines the properties of the microscopic system remains ambiguous: Are these properties "partially" actualized (see Dirac [34]), "potential" (see e.g. Popper [35] or Shimony [36]), "indeterminate" (see Reichenbach [37]) or "unsharp" (see e.g. Busch et al. [33])? Second, one faces then the "measurement problem". For this reason, a pragmatist approach as the one supported here should get ride of the notion of "state". As we will see below, it is possible to connect the state vector directly with the preparations without having recourse to the notion of physical state (the presence of the word "state" in the expression "vector state" should then be viewed as a historical accident).

Let us return to the notion of microscopic system and focus on a second pragmatic feature of it. When referring to a given microscopic system, say an "electron", the physicists implicitly think about a *set of equivalent preparations*, and not just to one particular preparation. For this reason, Peres writes [24, p. 24]:

> A quantum system is defined by an *equivalence class of preparations*. [...] For example, there are many equivalent macroscopic procedures for producing what we call a photon, or a free hydrogen atom, etc. The *equivalence* of different preparation procedures should be verifiable by suitable tests.

We can make this definition more precise by recalling a feature characterizing the set of preparations referring to a given microscopic system: each of these preparations leads to the *same numerical value* when the measurement of a "permanent" observable (e.g. mass, charge or spin) is performed. Therefore, I propose the following pragmatic definition for the expression "microscopic system":

***D*$_5$**: A "microscopic system", noted S, refers to a set of preparations leading to the same numerical value when the measurement of a permanent observable is performed.

By contrast with the instrumentalist interpretations mentioned in the introduction, this pragmatist approach has no intention to remove the expression "microscopic system" from the vocabulary used by the physicists in QM. It does not claim that QM

deals only with *macroscopic objects*, such as measurement apparatus. QM deals first of all with *microscopic systems*, it makes predictions for the measurements performed on those systems.

Nevertheless, what must be avoided from a pragmatist point of view is any metaphysical statement about these microscopic systems, such as: "They are autonomous systems", "They exist in the world independently of us"... This is not to say that the microscopic systems do not exist. But, the assumption of their existence has a meaning only within the frame of the research activity of microphysics. This assumption contributes to the success of this research activity. In this sense, it is a "sustainable" assumption.

Furthermore, what can legitimately (i.e. scientifically) be said about the microscopic systems should not go beyond the knowledge established in the frame of the research activity in microphysics. For, all the additional ontological representations associated to those systems (such as the particle and wave representations) have problematic consequences. On this point, pragmatism can be viewed as a kind of *internal realism*, i.e. "internal" to the physicists' practice.[8]

2.4 Measurement Outcome

We now have defined all the terms making it possible to refer to a measurement outcome in microphysics. Indeed, it corresponds typically to *an observable on a microscopic system, at a given time, having a certain numerical value*.

2.5 Composed System

If N distinct preparations P_1, $P_2, \ldots$, and P_N, producing respectively the microscopic systems S_1, $S_2, \ldots$, and S_N, are performed simultaneously, we can consider that these systems form a "composed system" noted $S = S_1 + S_2 + \cdots + S_N$. It is also possible to perform a unique preparation, noted P, of which the potential observable consequences are identical to the N preparations P_1, $P_2, \ldots$, and P_N performed simultaneously. From this, we can propose the following definition:

D_6: A "composed system" $S = S_1 + S_2 + \cdots + S_N$, consists of N microscopic systems S_1, $S_2, \ldots$, and S_N, which are produced either individually by N distinct preparations performed simultaneously, or jointly by a unique preparation.

We usually say: S_1, $S_2, \ldots$, and S_N are the "sub-systems" of S. Note that this definition is pragmatic insofar as the terms used to formulate it are themselves defined pragmatically.

2.6 Compatible and Incompatible Observables

The pragmatic definitions given above could as well fit for the pragmatist description of the research activity based on classical physics ("microscopic system" should simply be replaced by "macroscopic system"). What makes the specificity of the

[8] We refer here to the view supported by Hilary Putnam in the 1980's (see e.g. Putnam [38]).

research activity in microphysics is the existence of a relation of "incompatibility" between some observables. The compatibility and incompatibility of observables can be defined pragmatically as follows:

D_7: Two observables A and B are "compatible" *iff* when repeating a large number of times (tending ideally to infinite) the experiment which consists in measuring consecutively A, B, and A on a microscopic system S, the two measurements of A always provide the same outcome.

D_8: Two observables A and B are "incompatible" *iff* when repeating a large number of times (tending ideally to infinite) the experiment which consists in measuring consecutively A, B, and A on a microscopic system S, the two measurements of A do not always provide the same outcome.

3 Second Step: The Pragmatic Functions

Let us turn now to the second step of the pragmatist interpretation of QM and try to determine the functions any theory must fulfil so as to ensure the success of the research activity in microphysics. These functions can be called "pragmatic" since they are strictly relative to the physicists' practice. The aim is to describe what the physicists are really *doing* with a theory in the context of microphysics and not to make metaphysical speculations concerning the world as it may be in accordance with this theory. In order to avoid any intrusion of metaphysics, we will express these pragmatic functions only with the terms having received a pragmatic definition in the preceding section.

As stressed in Sect. 2.1, by making reference to a certain preparation, the physicists can pay attention directly to the potential observable consequences of the experimental operations they are performing, without describing the latter in detail. Now, there is an *infinite* number of distinct measurements that can be performed after a given preparation. Therefore, the set of potential observable consequences characterizing a preparation is itself *infinite*. So in practice, this set cannot be specified *explicitly*.

For this reason, if the physicists want to refer unambiguously to the same preparation, they will need a theoretical tool describing it synthetically, this is to say, a unified theoretical tool which encapsulates all its potential observable consequences. Here lies the first pragmatic function:

F_1: For any preparation P, the theory must enable the physicists to characterize it synthetically.

After having performed a preparation, the physicists can perform any kind of measurements. (As we just saw, there is an infinite number of possible measurements.) What does it mean then for a theory *to ensure the success of the research activity in microphysics*? Its means essentially: *to predict as well as possible the consequences of the experimental operations performed by the physicists in this context*. Indeed, by anticipating what happens in given experimental conditions, the physicists can have a better grasp on the physical world. For instance, they can develop new technologies.

Consider the measurement of a certain observable after a given preparation. To predict as well as possible what will happen consists, first, in determining the possible outcomes of this measurement, i.e. the possible numerical values this observable can have at the end of the measurement. So a second pragmatic function can be pointed out:

$\boldsymbol{F_2}$: For any observable A, the theory must enable the physicists to determine the set of possible numerical values that can be obtained when measuring A.

Second, to predict as well as possible what will happen consists in predicting *which* of these possible numerical values will occur as a matter of fact. Here, an *empirical constraint* proper to microphysics enters into play: there are observables being incompatible; this relation of incompatibility prevents the physicists from making deterministic predictions. For instance, when the measurement of an observable A on a microscopic system S yields a certain numerical value a_k (this measurement being equivalent to a certain preparation) and the measurement of an incompatible observable B on S is performed consecutively, then different numerical values $\{b_i\}$ can be obtained (although the preparation is exactly the same). More generally, there is no univocal connexion between a preparation and the outcome of a consecutive measurement.

Nevertheless, there are *empirical regularities*: the relative frequency of occurrence for each possible numerical value of a measurement consecutive to a given preparation stabilizes itself around a fix value when the same experience is repeated a large number of times (tending ideally to infinite). Now, this stabilized relative frequency can be converted into a *probability*. This (empirically established) probability constitutes a probabilistic prediction for any further experiment. Why couldn't we demand from a theory to determine this probability in an *a priori* manner? This brings to light the following pragmatic function:

$\boldsymbol{F_3}$: For any preparation P and for any observable A, the theory must enable the physicists to determine the probability of occurrence of each numerical value that can be obtained when measuring A consecutively to P.

As mentioned above, some preparations are producing composed systems. This calls for a fourth pragmatic function:

$\boldsymbol{F_4}$: The theory must fulfil the functions F_1 to F_3 when the preparation produces a composed system.

In this section, we have only considered the case where the measurement is consecutive to the preparation. But of course, in most experiments in microphysics, there is an intermediate phase between the preparation and the measurement. Therefore, the following pragmatic function has to be spelled out:

$\boldsymbol{F_5}$: The theory must fulfil the functions F_2 to F_4 when there is an intermediate phase.

Consider now an experiment where the physicists perform a sequence of experimental operations, and know that it corresponds to one of the preparations $\{P_i\}$, without knowing precisely which one. By definition, they have an incomplete knowledge

of the preparation (see Sect. 2.1). This incomplete knowledge expresses an uncertainty which can lead to a probabilistic judgment: being aware of some experimental parameters, the physicists can estimate that the sequence of experimental operations they have performed corresponds to preparation P_1 with probability p_1, to preparation P_2 with probability $p_2, \ldots$, and to preparation P_N with probability p_N. With respect to such a situation, a further pragmatic function has to be made explicit:

F_6: When the physicists have only an incomplete knowledge of the preparation, the theory must fulfil the functions F_1 to F_5 *modulo* the uncertainty introduced by this incomplete knowledge.

Finally, consider a measurement M which can produce a set of outcomes $\{o_i\}$. Each couple (M, o_i)—referring to M producing o_i—is equivalent to a different preparation. As a consequence, the knowledge of the measurement outcome constitutes essential information for the predictions concerning further possible measurements. A last pragmatic function must be put forward:

F_7: The theory must enable the physicists to incorporate the empirical knowledge gained from a measurement, so that the functions F_1 to F_6 can still be fulfilled after this measurement.

4 Third Step: Showing that QM Fulfills Exactly the Pragmatic Functions

The third step consists in showing that QM is the only theory that fulfils exactly the seven pragmatic functions formulated above. For this purpose, we will reconstruct the theory by starting merely with a real vector space with scalar product, and then by adding one by one each component of the theory. The usefulness of each of these components will be brought to light. That is, we will see how they enable to fulfil the different pragmatic functions. Our task is also to verify that any modification of QM implies either that the seven pragmatic functions are not all fulfilled, or that the theory entails a formal structure which is useless for the physicists' practice. Because of the infinite number of possible modifications of QM, this second task cannot in principle be exhaustively achieved. In this paper, we will consider only a limited number of modifications, letting open the program for a more systematic investigation.

4.1 Real Vector Space with Scalar Product

Consider a preparation P producing a microscopic system S, and an observable A which can be measured on S. Suppose the physicists have a certain *empirical* knowledge concerning P and A: they know the spectrum $\{a_i\}$, with $i = 1, \ldots, N$, of the values which can be obtained when measuring A on S, and they know the probabilities[9] $\{p(a_i)\}$ of occurrence of these values when A is measured on S consecutively to P. The values $\{a_i\}$ being mutually exclusive (i.e. they cannot occur simultaneously

[9]The probability of occurrence of an event e can be determined empirically by repeating the same experiment a large number of times (tending ideally to infinite). This probability is then defined *a posteriori* as being equal to the relative frequency of occurrence of e.

when measuring A), and jointly exhaustive (i.e. they cover the whole set of possible outcomes of the measurement of A), we have $\sum_{i=1}^{N} p(a_i) = 1$, in accordance with Kolmogorovian probability theory.

It is possible (although not necessary) to make use of a *real vector space of dimension N with a scalar product* in order to express synthetically the sets $\{a_i\}$ and $\{p(a_i)\}$. Let us note V_R this real vector space, and $(\mathbf{u}, \mathbf{v})$, with $\mathbf{u}, \mathbf{v} \in V_R$, the scalar product. The whole set $\{a_i\}$ can be encapsulated in a single linear operator $\hat{A} \equiv \sum_{i=1}^{N} a_i \hat{P}_{\mathbf{v}_i}$, where $\{\mathbf{v}_i\}$ is an orthonormal basis of V_R, and $\hat{P}_{\mathbf{v}_i}$ is the projector defined by equation $\hat{P}_{\mathbf{v}_i}\mathbf{v} \equiv (\mathbf{v}_i, \mathbf{v})\mathbf{v}_i$. According to the usual way of expression, operator $\hat{A}$ "represents" observable A. Whereas the whole set $\{p(a_i)\}$ can be encapsulated in a single vector $\mathbf{v}^S \equiv \sum_{i=1}^{N} \sqrt{p(a_i)}\mathbf{v}_i$. Let us stress two important points. First, each basis vector $\mathbf{v}_i$ is associated to a possible value a_i of A—this value being derived from the eigenvalue equation $\hat{A}\mathbf{v}_i = a_i\mathbf{v}_i$. Second, the square of the coefficient associated to each basis vector $\mathbf{v}_i$ in the expression of $\mathbf{v}^S$ can be used to represent the probability $p(a_i)$ of occurrence of the value a_i associated to $\mathbf{v}_i$—this probability being derived from $\mathbf{v}^S$ by equation $p(a_i) = (\mathbf{v}_i, \mathbf{v}^S)^2$, which corresponds to a restricted version of Born rule suited for a real vector space.

Consider now any observable B which is *incompatible* with A.[10] Let us note $\{b_i\}$ the values which can be obtained when measuring B on S, and $\{p(b_i)\}$ the probabilities of occurrence of these values when B is measured on S consecutively to P. Suppose the physicists have no empirical knowledge concerning the values $\{b_i\}$ and the probabilities $\{p(b_i)\}$. The main interest of a vector space formalism is that it makes it possible to derive, in an *a priori* manner, these probabilities $\{p(b_i)\}$ from the knowledge of the probabilities $\{p(a_i)\}$. Let us try to explain how.

It is essential first to emphasize the *symmetry relations* between the reference frames associated to the experimental devices suited for the measurement of incompatible observables. Indeed, these reference frames appear to be equivalent with respect to symmetries in the physical space-time (symmetries of rotation, of displacement...). As a simple example, consider the symmetry relation between the reference frames associated respectively to the experimental device suited for the measurement on an electron E of $S_{\mathbf{z}}$ (i.e. the spin component in direction $\mathbf{z}$) and the one suited for the measurement on E of $S_{\mathbf{a}}$ (i.e. the spin component in direction $\mathbf{a}$ which lies in the plane $O\mathbf{xz}$ and forms any angle θ with $\mathbf{z}$ (see Fig. 2)): these reference frames are equivalent with respect to a rotation of angle θ in the physical space.[11]

Now, from the existence of these symmetry relations we can infer that *incompatible observables have the same set of possible values*. As Leslie Ballentine [26, p. 63] points out, observables "differing only by transformation to another frame of reference [...] must have the same set of possible values". In other words, the set of values

[10] In the case of an observable B which is *compatible* with A, one can represent synthetically the sets of its possible values and corresponding probabilities of occurrence by appealing to a real vector space of dimension equal to the number of possible values of A plus the number of possible values of B. We won't develop this point further, since it is not needed to understand the essential usefulness of the vector space formalism in microphysics.

[11] The case of position and momentum is less obvious and will not be discussed here. On this point, see Jordan [39, 40] and [26, p. 78].

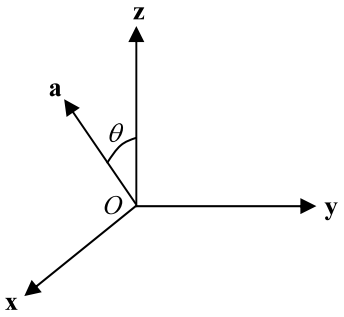

Fig. 2 Representation of the physical space, with **a** lying in the plane $O\mathbf{xz}$

$\{b_i\}$ is the same as the set of values $\{a_i\}$. For instance, if the physicists know from experience that the possible values which can be obtained when measuring $S_{\mathbf{z}}$ on E are $\{+\hbar/2, -\hbar/2\}$, they can infer that the possible values which can be obtained when measuring $S_{\mathbf{a}}$ on E are also $\{+\hbar/2, -\hbar/2\}$. Hence, the physicists can fulfil the pragmatic function F_2 merely by means of symmetry considerations.

The further essential point to be emphasized is the existence of an infinite number of orthonormal basis of V_R which are equivalent under the action of orthogonal operators. Each orthonormal basis makes it possible to express synthetically the possible values and corresponding probabilities of occurrence associated to a distinct observable. For instance, consider an orthonormal basis $\{\mathbf{w}_i\}$, which is equivalent to $\{\mathbf{v}_i\}$ under the action of a certain orthogonal operator $\hat{U}$—each $\mathbf{w}_i$ being derived from $\{\mathbf{v}_i\}$ by the relation $\mathbf{w}_i = \sum_{j=1}^{N} u_{ji}\mathbf{v}_j$, with $\{u_{ji}\}$ the real numbers characterizing $\hat{U}$. In order to represent the incompatible observable B and express synthetically the set of its possible values $\{b_i\}$, we can define the linear operator $\hat{B} \equiv \sum_{i=1}^{N} b_i \hat{P}_{\mathbf{w}_i}$, where $\hat{P}_{\mathbf{w}_i}$ is the projector defined by equation $\hat{P}_{\mathbf{w}_i}\mathbf{v} \equiv (\mathbf{w}_i, \mathbf{v})\mathbf{w}_i$. More important, one can imagine that $\hat{U}$ is such that the set $\{p(b_i)\}$ can be encapsulated in the same vector $\mathbf{v}^S$ as above, but decomposed on the basis $\{\mathbf{w}_i\}$ as follows: $\mathbf{v}^S = \sum_{i=1}^{N} \sqrt{p(b_i)}\mathbf{w}_i$.

Let us put forward a fundamental feature of the vector space formalism which makes this reformulation of $\mathbf{v}^S$ (in terms of the basis $\{\mathbf{w}_i\}$) acceptable from the point of view of Kolmogorovian probability theory: it enables the application of a generalised version of the Pythagorean Theorem.[12] Indeed, since the scalar product is linear (by definition) and the basis $\{\mathbf{v}_i\}$ and $\{\mathbf{w}_i\}$ are orthonormal, it follows that $(\mathbf{v}^S, \mathbf{v}^S) = \sum_{i=1}^{N} p(a_i)$ and $(\mathbf{v}^S, \mathbf{v}^S) = \sum_{i=1}^{N} p(b_i)$. These two equations and the equality $\sum_{i=1}^{N} p(a_i) = 1$ imply $\sum_{i=1}^{N} p(b_i) = 1$, in accordance with Kolmogorovian probability theory.

It must be stressed moreover that $\mathbf{v}^S = \sum_{i=1}^{N} \sqrt{p(a_i)}\mathbf{v}_i = \sum_{i=1}^{N} \sqrt{p(b_i)}\mathbf{w}_i$ iff the real numbers $\{u_{ji}\}$ are such that, for every i, $\sqrt{p(b_i)} = \sum_{j=1}^{N} \sqrt{p(a_j)}u_{ji}$. It appears therefore that if we know how to transform appropriately the basis $\{\mathbf{v}_i\}$ into the basis $\{\mathbf{w}_i\}$, i.e. if we know the appropriate orthogonal operator $\hat{U}$, we are able to derive, in an *a priori* manner, probabilities $\{p(b_i)\}$ from the knowledge of probabilities $\{p(a_i)\}$.

How can we determine the appropriate orthogonal operator $\hat{U}$? Here the symmetry relations between the reference frames associated to the experimental devices suited

[12] Both Hughes [23, pp. 83–84] and van Fraassen [2, p. 115] have discussed the importance of this generalised version of the Pythagorean Theorem in the frame of the Hilbert space formalism.

for the measurement of incompatible observables come into play a second time. For, these *symmetry relations in the physical space-time can be put in correspondence with symmetry relations in a vector space*. For instance, in the case of observables $S_{\mathbf{z}}$ and $S_{\mathbf{a}}$, the above mentioned rotation symmetry of angle θ in the physical space can be put in correspondence with a rotation symmetry of angle $\theta/2$ in a real vector space.

Let us make this point explicit. For simplicity, we will note $\{+,-\}$ the possible values $\{+\hbar/2,-\hbar/2\}$ of $S_{\mathbf{z}}$. Since there are only two possible values, the real vector space V_R can be defined has being bi-dimensional. In order to represent $S_{\mathbf{z}}$ and express synthetically the set of its possible values $\{+,-\}$, we can define the linear operator $\hat{S}_{\mathbf{z}} \equiv +\hat{P}_{\mathbf{v}_+} - \hat{P}_{\mathbf{v}_-}$, where $\hat{P}_{\mathbf{v}_+}$ and $\hat{P}_{\mathbf{v}_-}$ are the projectors defined respectively by $\hat{P}_{\mathbf{v}_+}\mathbf{v} \equiv (\mathbf{v}_+,\mathbf{v})\mathbf{v}_+$ and $\hat{P}_{\mathbf{v}_-}\mathbf{v} \equiv (\mathbf{v}_-,\mathbf{v})\mathbf{v}_-$, with $\{\mathbf{v}_+,\mathbf{v}_-\}$ an orthonormal basis of V_R.

Suppose the physicists know the probabilities $\{p(+),p(-)\}$ of occurrence of $\{+,-\}$ when $S_{\mathbf{z}}$ is measured on E consecutively to a preparation P. These probabilities can be encapsulated in the vector $\mathbf{v}^E \equiv \sqrt{p(+)}\mathbf{v}_+ + \sqrt{p(-)}\mathbf{v}_-$.

Similarly, in order to represent $S_{\mathbf{a}}$ and express synthetically the set of its possible values $\{+,-\}$, we can define the linear operator $\hat{S}_{\mathbf{a}} \equiv +\hat{P}_{\mathbf{v}^{\mathbf{a}}_+} - \hat{P}_{\mathbf{v}^{\mathbf{a}}_-}$, where $\hat{P}_{\mathbf{v}^{\mathbf{a}}_+}$ and $\hat{P}_{\mathbf{v}^{\mathbf{a}}_-}$ are the projectors defined respectively by $\hat{P}_{\mathbf{v}^{\mathbf{a}}_+}\mathbf{v} \equiv (\mathbf{v}^{\mathbf{a}}_+,\mathbf{v})\mathbf{v}^{\mathbf{a}}_+$ and $\hat{P}_{\mathbf{v}^{\mathbf{a}}_-}\mathbf{v} \equiv (\mathbf{v}^{\mathbf{a}}_-,\mathbf{v})\mathbf{v}^{\mathbf{a}}_-$, with $\{\mathbf{v}^{\mathbf{a}}_+,\mathbf{v}^{\mathbf{a}}_-\}$ another orthonormal basis of V_R.

Yet, the rotation symmetry of angle θ in the physical space between the reference frames associated respectively to the experimental device suited for the measurement of $S_{\mathbf{z}}$ and the one suited for the measurement of $S_{\mathbf{a}}$ can be put in correspondence with a rotation symmetry of angle $\theta/2$ in V_R between the basis $\{\mathbf{v}_+,\mathbf{v}_-\}$ and $\{\mathbf{v}^{\mathbf{a}}_+,\mathbf{v}^{\mathbf{a}}_-\}$. That is to say, the basis $\{\mathbf{v}^{\mathbf{a}}_+,\mathbf{v}^{\mathbf{a}}_-\}$ can be derived from $\{\mathbf{v}_+,\mathbf{v}_-\}$ by the action of a rotation operator[13] of angle $\theta/2$:

$$\mathbf{v}^{\mathbf{a}}_+ = \cos\theta/2\mathbf{v}_+ + \sin\theta/2\mathbf{v}_-$$
$$\mathbf{v}^{\mathbf{a}}_- = -\sin\theta/2\mathbf{v}_+ + \cos\theta/2\mathbf{v}_-$$

This makes it possible to derive, in an *a priori* manner, the probabilities $\{p^{\mathbf{a}}(+),p^{\mathbf{a}}(-)\}$ of occurrence of $\{+,-\}$ when $S_{\mathbf{a}}$ is measured on E consecutively to P from the knowledge of probabilities $\{p(+),p(-)\}$. For this purpose, $\mathbf{v}^E$ must be rewritten on the basis $\{\mathbf{v}^{\mathbf{a}}_+,\mathbf{v}^{\mathbf{a}}_-\}$:

$$\mathbf{v}^E = \left(\sqrt{p(+)}\cos\theta/2 + \sqrt{p(-)}\sin\theta/2\right)\mathbf{v}^{\mathbf{a}}_+ + \left(-\sqrt{p(+)}\sin\theta/2 + \sqrt{p(-)}\cos\theta/2\right)\mathbf{v}^{\mathbf{a}}_-$$

We can then equate probabilities $\{p^{\mathbf{a}}(+),p^{\mathbf{a}}(-)\}$ to the square of the corresponding coefficients of $\mathbf{v}^E$ as follows:

$$p^{\mathbf{a}}(+) = \left(\sqrt{p(+)}\cos\theta/2 + \sqrt{p(-)}\sin\theta/2\right)^2$$

[13] A rotation operator acting on a real vector space is always orthogonal.

$$p^{\mathbf{a}}(-) = \left(-\sqrt{p(+)}\sin\theta/2 + \sqrt{p(-)}\cos\theta/2\right)^2$$

The restricted version of Born rule suited for a real vector space is here applied.

This example illustrates how the symmetry relations in the physical space-time can be translated into the vector space formalism, and how, given the probabilities $\{p(a_i)\}$ relative to a certain observable A, this vector space formalism enables to derive, in an *a priori* manner, the probabilities $\{p(b_i)\}$ relative to an observable B which is incompatible with A.

In other words, this discussion shows how a real vector space with a scalar product and an appropriate rule for the derivation of probabilities (Born rule) enables to fulfil the pragmatic function F_3 for a wide range of cases (although not for all cases as we will see in the next subsection).

Furthermore, it appears that a single vector $\mathbf{v}^S$, insofar as it is decomposable on an infinite number of different orthonormal basis, can encapsulate the probabilities $\{p(a_i)\}, \{p(b_i)\}, \ldots$ relative to all the observables $A, B, \ldots$ which can be measured on a microscopic system S consecutively to a given preparation P. That is to say, a vector can play the role of a unified theoretical tool expressing all the potential observable consequences of a given preparation producing a certain microscopic system. It can therefore fulfil the pragmatic function F_1. In QM, such a vector is commonly called "state vector".[14]

4.2 Complex Numbers

Why is QM based not merely on real vector spaces but on *complex* vector spaces? What is the practical usefulness of complex numbers? A simple answer can be given by discussing further the case of the measurement of the different spin-components on an electron. Until now, we have been concerned only with the derivation of probabilities for an observable $S_{\mathbf{a}}$, with $\mathbf{a}$ lying in the plane $O\mathbf{xz}$ and forming any angle θ with $\mathbf{z}$. A bi-dimensional real vector space formalism is adequate in order to deal with this case. But it is unable to derive the probabilities for an observable $S_{\mathbf{d}}$, with $\mathbf{d}$ being any direction of the physical space, i.e. forming any angle θ with $\mathbf{z}$ in the plane $O\mathbf{xz}$ and forming any angle ϕ with $\mathbf{x}$ in the plane $O\mathbf{xy}$ (see Fig. 3).

The reason of this failure is that a bi-dimensional real vector space can account only for one degree of freedom (for instance the one associated to θ) and not for two (i.e. the ones associated to θ and ϕ), as requested here.

Yet, by introducing the complex numbers into the vector space formalism, it becomes possible to account for both degrees of freedom. This *connexion between complex numbers and degrees of freedom* has been put forward by David Finkelstein [29, pp. 59–60], and can be found (although not explicitly) in R.I.G. Hughes' book on the interpretation of QM [23, Chap. 4].

[14]The introduction of the term "state" in QM is due to Dirac [34]. Since the expression "state vector" is the one commonly used in QM, we will maintain it in the next sub-sections. This will prevent us from any complication or confusion. However, we will associate to this expression only a pragmatic (or operational) meaning, the one being exposed here. Any ontological interpretation (see the discussion in Sect. 2.3) will be put aside.

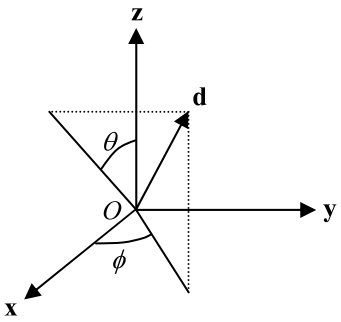

Fig. 3 Representation of the physical space, with **d** being in any direction

To explain this connexion, let us first recall that a complex number c is composed of two real numbers a and b, and can be written $c = a + ib$, where i is an imaginary coefficient. Consider the following changes: V_R is replaced by a complex vector space V_C, the former basis $\{\mathbf{v}_+, \mathbf{v}_-\}$ is redefined on V_C, and $\hat{S}_\mathbf{z}$ and $\mathbf{v}^E$ are redefined in terms of this basis like previously.

In order to represent $S_\mathbf{d}$ and express synthetically the set of its possible values $\{+, -\}$, we can define the linear operator $\hat{S}_\mathbf{d} \equiv +\hat{P}_{\mathbf{v}^\mathbf{d}_+} - \hat{P}_{\mathbf{v}^\mathbf{d}_-}$, where $\hat{P}_{\mathbf{v}^\mathbf{d}_+}$ and $\hat{P}_{\mathbf{v}^\mathbf{d}_-}$ are the projectors defined respectively by $\hat{P}_{\mathbf{v}^\mathbf{d}_+}\mathbf{v} \equiv (\mathbf{v}^\mathbf{d}_+, \mathbf{v})\mathbf{v}^\mathbf{d}_+$ and $\hat{P}_{\mathbf{v}^\mathbf{d}_-}\mathbf{v} \equiv (\mathbf{v}^\mathbf{d}_-, \mathbf{v})\mathbf{v}^\mathbf{d}_-$, with $\{\mathbf{v}^\mathbf{d}_+, \mathbf{v}^\mathbf{d}_-\}$ another orthonormal basis of V_C.

How can we derive the probabilities $\{p^\mathbf{d}(+), p^\mathbf{d}(-)\}$ of occurrence of $\{+, -\}$ when $S_\mathbf{d}$ is measured on E consecutively to P? The method is similar as above. The basis $\{\mathbf{v}^\mathbf{d}_+, \mathbf{v}^\mathbf{d}_-\}$ is derived from $\{\mathbf{v}_+, \mathbf{v}_-\}$ by the action of a rotation operator[15] as follows:

$$\mathbf{v}^\mathbf{d}_+ = \cos\theta/2 e^{-i\phi/2}\mathbf{v}_+ + \sin\theta/2 e^{i\phi/2}\mathbf{v}_-$$
$$\mathbf{v}^\mathbf{d}_- = -\sin\theta/2 e^{-i\phi/2}\mathbf{v}_+ + \cos\theta/2 e^{i\phi/2}\mathbf{v}_-$$

where $e^{i\delta}$ is a complex number equal to $\cos\delta + i\sin\delta$. On this new basis, $\mathbf{v}^E$ is rewritten:

$$\mathbf{v}^E = \left(\sqrt{p(+)}\cos\theta/2 e^{i\phi/2} + \sqrt{p(-)}\sin\theta/2 e^{-i\phi/2}\right)\mathbf{v}^\mathbf{d}_+ + \left(-\sqrt{p(+)}\sin\theta/2 e^{i\phi/2} + \sqrt{p(-)}\cos\theta/2 e^{-i\phi/2}\right)\mathbf{v}^\mathbf{d}_-$$

We can then equate probabilities $\{p^\mathbf{d}(+), p^\mathbf{d}(-)\}$ to the square modulus of the corresponding coefficients of $\mathbf{v}^E$ as follows:

$$p^\mathbf{d}(+) = \left|\sqrt{p(+)}\cos\theta/2 e^{i\phi/2} + \sqrt{p(-)}\sin\theta/2 e^{-i\phi/2}\right|^2$$
$$p^\mathbf{d}(-) = \left|-\sqrt{p(+)}\sin\theta/2 e^{i\phi/2} + \sqrt{p(-)}\cos\theta/2 e^{-i\phi/2}\right|^2$$

[15] A rotation operator acting on a complex vector space is always unitary—a unitary operator being equivalent to an orthogonal operator on real vector space.

Taking the square of the modulus secures the derived probabilities to be real numbers, in accordance with Kolmogorovian probability theory. This is the standard version of Born rule suited for a complex vector space.

At this point, we could imagine modifying QM by having recourse to vector spaces defined on quaternions. Recall that a quaternion q is composed of four real numbers a, b, c and d, and can be written $q = a + ib + jc + kd$, where i, j and k are imaginary coefficients. Hence, a vector defined on a quaternionic vector space could account for two supplementary degrees of freedom. However, this additional mathematical power does not help to make any new predictions in microphysics.[16] So, such a modification of QM would simply be useless from a practical point of view.

4.3 Born Rule

Among the various recent deductions of Born rule, the one proposed by Scott Aaronson [42] appears to be appropriate for a pragmatist interpretation.[17] Let us return to the general example discussed in Sect. 4.1 and describe it by means of a complex vector space. In the frame of QM, $\mathbf{v}^S = \sum_{i=1}^{N} c_i \mathbf{v}_i$ (the $\{c_i\}$ being now complex numbers) is usually constructed such that the probabilities $\{p(a_i)\}$, which are known from experience, can be derived from $\mathbf{v}^S$ by equation $p(a_i) = |(\mathbf{v}_i, \mathbf{v}^S)|^2$, which corresponds to Born rule. A possible modification of QM would be to construct $\mathbf{v}^S$ such that the probabilities $\{p(a_i)\}$ are derived from $\mathbf{v}^S$ by equation $p(a_i) = |(\mathbf{v}_i, \mathbf{v}^S)|^p$, with p different from 2. Is such a modification admissible?

To answer this question, recall that the aim of the complex vector space formalism is to derive, in an *a priori* manner, probabilities $\{p(b_i)\}$ from the knowledge of probabilities $\{p(a_i)\}$. Hence, the rule for the derivation of probabilities $\{p(a_i)\}$ from $\mathbf{v}^S$ must hold for the derivation of probabilities $\{p(b_i)\}$. In other words, if $p(a_i) = |(\mathbf{v}_i, \mathbf{v}^S)|^p$, then we must also have $p(b_i) = |(\mathbf{w}_i, \mathbf{v}^S)|^p$.

Recall from Sect. 4.1 that we want the basis $\{\mathbf{w}_i\}$ to be orthonormal. This implies that $\{\mathbf{w}_i\}$ is derived from $\{\mathbf{v}_i\}$ by the action of a *unitary* operator, say the operator $\hat{U}$ characterized by the complex numbers $\{u_{ji}\}$ such that $\mathbf{w}_i = \sum_{j=1}^{N} u_{ji} \mathbf{v}_j$. Accordingly, $p(b_i)$ can be rewritten $p(b_i) = |\sum_{j=1}^{N} c_j u_{ji}|^p$.

Yet, Kolmogorovian probability theory imposes the constraint $\sum_{i=1}^{N} p(b_i) = 1$, or if we replace $p(b_i)$ with the above expression:

$$\sum_{i=1}^{N} \left| \sum_{j=1}^{N} c_j u_{ji} \right|^p = 1$$

Aaronson's strategy consists then in choosing a certain $c_k \neq 0$ and in replacing it by $e^{i\varphi} c_k$, where φ is a real number, so that $\mathbf{v}^S$ is replaced by $\tilde{\mathbf{v}}^S = e^{i\varphi} c_k \mathbf{v}_i + \sum_{\substack{i=1 \\ i \neq k}}^{N} c_i \mathbf{v}_i$. Note that this change has no consequence for the derivation rule of $p(a_i)$,

[16] Some generalisations of QM to quaternionic vector spaces have been proposed. See for instance the one of Stephen Adler [41]. This author explicitly admits that his quaternionic QM does not provide new predictions which could be experimentally tested (Ibid., p. 526)

[17] Alternative deductions have been proposed by Deutsch [43], Zurek [44] and Saunders [45].

since $p(a_k) = |(\mathbf{v}_k, \tilde{\mathbf{v}}^S)|^p = |e^{i\varphi} c_k|^p = |c_k|^p$. With this change, the preceding constraint becomes:

$$\sum_{i=1}^{N} \left| e^{i\varphi} c_k u_{ki} + \sum_{\substack{j=1 \\ j\neq k}}^{N} c_j u_{ji} \right|^p = 1$$

Now, Aaronson shows that this equation holds iff $p = 2$. Indeed, if $p \neq 2$, then we should have, for all i, either that $u_{ki} = 0$ or that $\sum_{\substack{j=1 \\ j\neq k}}^{N} c_j u_{ji} = 0$, implying that $\hat{U}$ is not unitary, contradicting one of the initial assumptions.

4.4 Completeness and Separability

QM is based on what von Neumann [46] coined "Hilbert spaces", that is to say, complex vector spaces (with a scalar product) which are complete and separable.

Why should the physicists appeal to *complete* vector spaces? This question has a simple practical answer: the property of completeness is required when the physicists are dealing with an observable A having an *infinite* number of possible values $\{a_i\}$. In such a case, obviously the vector space has to be defined as infinite-dimensional. More precisely, consider the complex vector space (with a scalar product) V_C defined by the orthonormal basis $\{\mathbf{v}_i\}$, with $i = 1, \ldots, \infty$. The problem is that the vector $\mathbf{v}^S = \sum_{i=1}^{\infty} \sqrt{p(a_i)}\mathbf{v}_i$, intended to express synthetically the probabilities $\{p(a_i)\}$ of occurrence of the values $\{a_i\}$, corresponds to an infinite sequence of vectors, and for this reason, does not belong to V_C (on this point, see [26, p. 26]). The solution is to appeal to the completion of V_C. Indeed, this completion entails all the infinite sequences of vectors having a finite norm. Now, the fact that $\sum_{i=1}^{\infty} p(a_i) = 1$ implies that $\mathbf{v}^S = \sum_{i=1}^{\infty} \sqrt{p(a_i)}\mathbf{v}_i$ has a finite norm, and thus, belongs to the completion of V_C.

As for the property of *separability*, its usefulness is less obvious. It seems that von Neumann's main motivation for the introduction of separability was mathematical rigour (see Barrett [47]). Nowadays, there is still no consensus on the necessity of adding this property in quantum mechanics. For this reason, we won't discuss this point any further.

From now on, we will adopt the usual notation of QM (introduced by Dirac [48]), that is to say, a preparation P producing a microscopic system S will be characterized by a state vector noted $|\psi^S\rangle$ defined on a Hilbert space H^S. The scalar product $(|\psi\rangle, |\phi\rangle)$ between two vectors $|\psi\rangle$ and $|\phi\rangle$ of H^S will be expressed $\langle\psi|\phi\rangle$, where $\langle\psi|$ is the dual vector of $|\psi\rangle$.

4.5 Tensor Product

Consider a preparation P producing a microscopic system S composed of two microscopic sub-systems S_1 and S_2.[18] Let H^{S_1} and H^{S_2} be the Hilbert spaces associated respectively to S_1 and S_2. How should H^{S_1} and H^{S_2} be unified so as to get a Hilbert

[18]The following discussion can be generalised to the case of an arbitrarily large number of sub-systems.

space H^S associated to S enabling the physicists to determine the occurrence probabilities for the outcomes of the measurements that can be performed on S (i.e. on S_1 and/or S_2) consecutively to P, that is, enabling them to fulfil the pragmatic function F_4?

A very simple way to achieve this unification consists in applying the *tensor product* $\otimes$ (for its definition, see e.g. Jauch [49]), or more precisely, in defining H^S by means of the vectors $\{|\chi_i\rangle \otimes |\varphi_j\rangle\}$, where $\{|\chi_i\rangle\}$ and $\{|\varphi_i\rangle\}$ are basis respectively of H^{S_1} and H^{S_2}. The pragmatic function F_4 can be fulfilled by means of vectors and linear operators appropriately defined on this tensor product Hilbert space in a similar manner as in the previous sub-sections. There is therefore no need to go in any further detail here.

Are there other ways to unify H^{S_1} and H^{S_2} in order to fulfil F_4? Although this may be the case, the tensor product formalism used in QM seems to be the simpler option. It is likely that the alternative mathematical constructions (if they exist) supply a formal complexity which is useless for the physicists' practice. Nevertheless, a more detailed investigation is required to clarify this point.

4.6 Schrödinger's Equation

Consider a preparation P producing a microscopic system S. Suppose the physicists have a complete knowledge of the preparation they are performing. On this basis they associate the state vector $|\psi^S(t_1)\rangle$ to S at time t_1 (where t_1 corresponds to the end of P as in Fig. 1). This state vector enables them to make probabilistic predictions concerning any observable that could be measured on S at time t_1. Suppose however they intend to perform the measurement of an observable on S at a later time t_2. In other words, there is an intermediate phase I between the preparation and the measurement. Now, so as to fulfil the pragmatic function F_5, that is, so as to make the appropriate probabilistic predictions concerning this measurement, the physicists have to derive the appropriate state vector $|\psi^S(t_2)\rangle$ associate to S at time t_2. How can this be done? So as to answer this question, we will make several pragmatic assumptions. Taking advantage of the works of R.I.G. Hughes [23, pp. 114–117] and Michel Bitbol [25, pp. 266–268], we will try to show how the general form of *Schrödinger's equation* can be derived only from these assumptions.

Let us assume that the measurement outcome depends only on P and the experimental conditions characterizing I. This can be translated mathematically by $|\psi^S(t_2)\rangle = \hat{U}_{t_1\to t_2}|\psi^S(t_1)\rangle$, where $\hat{U}_{t_1\to t_2}$ is an operator acting on $|\psi^S(t_1)\rangle$ and taking account of the experimental conditions of I. Of course, this reasoning holds also if the measurement is performed at a later time t_3: $|\psi^S(t_3)\rangle = \hat{U}_{t_1\to t_3}|\psi^S(t_1)\rangle$. Note moreover that the delimitation between P, I, and the measurement is partly a matter of convention so that P and I taken together can be considered as a new preparation P^* characterized by $|\psi^S(t_2)\rangle$. If we apply the preceding reasoning once again, we obtain $|\psi^S(t_3)\rangle = \hat{U}_{t_2\to t_3}|\psi^S(t_2)\rangle$. Consequently $\hat{U}_{t_1\to t_3} = \hat{U}_{t_2\to t_3}\hat{U}_{t_1\to t_2}$. Generalizing this discussion to the case of four successive times t_1, t_2, t_3, and t_4, we can deduce that $\hat{U}_{t_1\to t_4} = (\hat{U}_{t_3\to t_4}\hat{U}_{t_2\to t_3})\hat{U}_{t_1\to t_2} = \hat{U}_{t_3\to t_4}(\hat{U}_{t_2\to t_3}\hat{U}_{t_1\to t_2})$.

Let us assume that the time is homogeneous and that the experimental conditions characterizing the intermediate phase do not evolve. It follows from these two assumptions that the action of $\hat{U}_{t_A\to t_B}$ depends only on the time interval $t_B - t_A$ and not

on t_A and t_B themselves. Thus, if we define $t \equiv t_2 - t_1$, $t' \equiv t_3 - t_2$, and $t'' \equiv t_4 - t_3$, the previous equation becomes $(\hat{U}_{t''}\hat{U}_{t'})\hat{U}_t = \hat{U}_{t''}(\hat{U}_{t'}\hat{U}_t)$. In other words, the composition of the operators $\{\hat{U}_t\}$ is associative.

In the case $t = 0$, it is natural to assume that $\hat{U}_0$ leaves the state vector unchanged, that is to say, corresponds to the identity operator $\hat{I}$. Hence, with respect to the compositions of the operators $\{\hat{U}_t\}$, $\hat{U}_0$ is a neutral element.

Unless one interprets the state vector as representing the physical state of the studied system and one believes that the latter can evolve in a irreversible way (such an ontological interpretation of QM is here put aside), nothing prevents us from assuming the reversibility of the action of $\hat{U}_t$, or in other words, the existence for any $\hat{U}_t$ of an inverse operator $\hat{U}_t^{-1}$ such that $\hat{U}_t^{-1}\hat{U}_t = \hat{U}_t\hat{U}_t^{-1} = \hat{I}$.

The property of associativity of the composition operation, the existence of a neutral element, and the existence for any $\hat{U}_t$ of an inverse operator implies that the set $\{\hat{U}_t\}$ can be identified as a group with one real parameter t.

Let us assume furthermore that the operators $\{\hat{U}_t\}$ act on the state vector $|\psi^S(t)\rangle$ so that the latter evolves continuously with t. Just like the reversibility assumption, this continuity assumption consists in fact in *not making* a strong assumption (motivated by an ontological interpretation of QM), namely that the physical state of the studied system represented by the state vector makes discontinuous quantum jumps. By assuming continuity, the set $\{\hat{U}_t\}$ can be described more exactly as a continuous group with one real parameter t.

A last point has to be stressed. The operators $\{\hat{U}_t\}$ must be such that the probabilities derived from the state vector $|\psi^S(t)\rangle$ concerning the measurement of any observable are in accordance with Kolmogorovian probability theory. In particular, they must be such that the sum of these probabilities equal to 1. This is the case only if the norm of $|\psi^S(t)\rangle$ remains the same (equal to 1) for any time t. Yet, only unitary or antiunitary operators conserve the norm of a vector. If we choose $\hat{U}_t$ as antiunitary for a certain t, then $\hat{U}_{2t} = \hat{U}_t\hat{U}_t$ would automatically be unitary (see for instance [39, pp. 97–98]). Thus, so as to be coherent, we must choose the operators $\{\hat{U}_t\}$ as being all unitary.

Now, Marshall Stone [50] has established the following theorem:[19] if the set $\{\hat{U}_t\}$ forms a continuous group of unitary operators with one real parameter t, then there is a unique Hermitian operator $\hat{A}$ such that $\hat{U}_t = e^{it\hat{A}}$. If we rewrite $\hat{A} \equiv -\frac{1}{\hbar}\hat{H}$, then $\hat{U}_t = e^{-\frac{i}{\hbar}t\hat{H}}$, from which we get $i\hbar\frac{d}{dt}|\psi^S(t)\rangle = \hat{H}|\psi^S(t)\rangle$. This equation of evolution has exactly the same form as Schrödinger's equation, i.e. the ordinary dynamical equation of QM.

What remains to establish for a full derivation of Schrödinger's equation is that the operator $\hat{H}$ is the "Hamiltonian" in the usual sense, that is, the operator representing the *energy* observable. According to Hughes [23, p. 115], "what such an investigation [i.e. an investigation based on pragmatist assumptions and making use of Stone's theorem] [does] not show is why this operator [i.e. $\hat{H}$] should be the Hamiltonian (the energy operator) for the system". Nonetheless, what can be shown is that $\hat{H}$ shares

[19] This theorem has been put forward by several authors. See for instance Mackey [51, 2, pp. 177–181, 23, pp. 113–118], and [39, pp. 49–53 and 98].

two features with the Hamitonian. First, in virtue of Stone's theorem, $\hat{A}$ is Hermitian, so $\hat{H}$ is also Hermitian. This means that $\hat{H}$ has the mathematical form of an operator representing an observable,[20] like the Hamiltonian. Second, $\hat{H}$ commutes with itself (this is trivial), and hence, with all the evolution operators $\{\hat{U}_t = e^{-\frac{i}{\hbar}t\hat{H}}\}$. As proved by van Fraassen [2, p. 182], "a quantity is conserved (remains invariant over time) exactly if its representing operator commutes with all the evolution operators". It follows that the quantity represented by $\hat{H}$ is conserved over time. So $\hat{H}$ is similar to the usual Hamiltonian also because of its ability to represent a time-invariant quantity.

This partial derivation of Schrödinger's equation brings into light its usefulness: it preserves the predictive power of the state vector in time. The interpretation of this equation as representing a law of nature (governing the evolution of the physical state of the studied system) appears superfluous. As Schrödinger [52] himself has emphasized, "what do change are the *statistics* or *probabilities*, *these* moreover causally".

Could the dynamical equation of QM be different and still fulfil the pragmatic function F_5? Two important modifications of this equation can be imagined. Suppose first that it does not entail the term $\frac{d}{dt}|\psi^S(t)\rangle$. In this case, the equation would no more be able to determine the evolution of $|\psi^S(t)\rangle$ in time. So this modification is obviously not satisfactory. Second, suppose the dynamical equation is *nonlinear*, that is to say, expressed as follows: $\frac{d}{dt}|\psi^S(t)\rangle = f(|\psi^S(t)\rangle)$, where $f(|\psi^S(t)\rangle)$ is a nonlinear function of $|\psi^S(t)\rangle$. This modification has been imagined in a general and abstract way by Steven Weinberg [53], and in a particular way in the spontaneous reduction theory (see [5, 6]). The main problem with such a modification is that it implies contradiction with special relativity theory (see Gisin [54] and Polchinski [55]).

What about adding a *second dynamical equation*? This has been proposed by the proponents of the pilot-wave theory (see [3, 4]). According to them, the additional equation is supposed to describe the evolution of the "intrinsic position" of the studied system. Yet, the postulation of this equation is only motivated by the desire of getting a certain representation of the world (conceived as strictly deterministic). It implies no new testable predictions. With respect to the physicists' practice, this additional equation is useless. It supplies a surplus formal structure.

4.7 Density Operators

Suppose the physicists have only an incomplete knowledge concerning the actual preparation they have performed. Suppose more exactly that, based on some knowledge of the experimental conditions, they estimate the preparation to be P_1 with probability p or preparation P_2 with probability $\tilde{p}$, such that $p + \tilde{p} = 1$. Suppose also that both P_1 and P_2 are producing the same microscopic system S. Let's take $|\psi^S\rangle$ and $|\tilde{\psi}^S\rangle$ as the state vectors characterizing respectively P_1 and P_2. Consider any observable A which can be measured on S consecutively to P_1 or to P_2. Let us represent it by the operator $\hat{A}$, with $\{|a_i\rangle\}$ its eigenvectors and $\{a_i\}$ its eigenvalues, such that the former vectors can be expressed $|\psi^S\rangle = \sum_{i=1}^N c_i|a_i\rangle$ and $|\tilde{\psi}^S\rangle = \sum_{i=1}^N \tilde{c}_i|a_i\rangle$, where $\{c_i\}$ and $\{\tilde{c}_i\}$ are complex numbers. By means of Born rule, the physicists can

[20] Note that the eigenvalues of a Hermitian operator are real (just like the possible values of an observable).

predict that the occurrence probability of the value a_i is $p_1(a_i) = |\langle a_i|\psi^S\rangle|^2 = |c_i|^2$ if the actual preparation is P_1, and $p_2(a_i) = |\langle a_i|\tilde{\psi}^S\rangle|^2 = |\tilde{c}_i|^2$ if the actual preparation is P_2. Taking into account the uncertainty concerning the actual preparation, the overall occurrence probability of a_i is $p(a_i) = pp_1(a_i) + \tilde{p}p_2(a_i) = p|\tilde{c}_i|^2 + \tilde{p}|\tilde{c}_i|^2$.

Now, one can well imagine a more complicated situation where N preparations are considered as being possibly the preparation which has been actually performed, with N a large and possibly infinite number. In this case, the calculation of the overall occurrence probability of a_i brings into play N state vectors, and becomes for this reason much more difficult to achieve. Isn't there a unified theoretical tool making it possible to simplify this calculation and hence to fulfil the pragmatic function F_6?

The solution which has been found in QM is to appeal to the *density operator formalism* (see e.g. [39, pp. 73–78]). In the previous simple case, the incomplete knowledge concerning the actual preparation can be represented by the density operator $\hat{\rho}^S = p\hat{P}_{|\psi^S\rangle} + \tilde{p}\hat{P}_{|\tilde{\psi}^S\rangle}$, where $\hat{P}_{|\psi^S\rangle}$ and $\hat{P}_{|\tilde{\psi}^S\rangle}$ are projectors defined by $\hat{P}_{|\psi^S\rangle} \equiv |\psi^S\rangle\langle\psi^S|$ and $\hat{P}_{|\tilde{\psi}^S\rangle} \equiv |\tilde{\psi}^S\rangle\langle\tilde{\psi}^S|$. The overall occurrence probability of a_i can then be determined by means of a generalized version of Born rule as follows: $p(a_i) = \mathrm{Tr}(\hat{P}_{|a_i\rangle}\hat{\rho}^S) = p|c_i|^2 + \tilde{p}|\tilde{c}_i|^2$. This density operator formalism can be applied for an arbitrarily large number of preparations considered as being possibly the actual preparation.

Couldn't we make such a simple calculation merely on the basis of a *vector* defined in an appropriate manner? Let us imagine two ways of constructing the required unified tool. Consider first the vector $|\chi^S\rangle = \sqrt{p}|\psi^S\rangle + \sqrt{\tilde{p}}|\tilde{\psi}^S\rangle$. It is easy to see that the application of Born rule does not provide the expected result: $p(a_i) = |\langle a_i|\chi^S\rangle|^2 = |\sqrt{p}c_i + \sqrt{\tilde{p}}\tilde{c}_i|^2 \neq p|c_i|^2 + \tilde{p}|\tilde{c}_i|^2$. Consider second the vector defined as follows: $|\varphi^S\rangle = \sum_{i=1}^{N}\sqrt{p|\tilde{c}_i|^2 + \tilde{p}|\tilde{c}_i|^2}|a_i\rangle$. Although we apparently get the expected result when applying Born rule to this vector, this is the case only for one specific observable, namely A. One can easily verify that for any observable incompatible with A, this vector leads to incorrect predictions.

4.8 Projection Postulate and Probability Conditionalisation Rule

The projection postulate is often identified with the idea that during a measurement the studied microscopic system is making a quantum jump, i.e. its physical state is instantaneously, indeterministically and irreversibly reduced.[21] Yet, this quantum jump hypothesis is unsatisfactory because of several problems. In particular, there is no criterion for determining when this alleged quantum jump occurs, and in the case of EPR-type experiments, this hypothesis implies non-local effects in contradiction with special relativity theory (see e.g. [20]). Being assumed as identical, both the quantum jump hypothesis and the projection postulate are rejected by some interpretations of QM, e.g. by the many-worlds, many-minds and modal interpretations (see [1, 2, 7–14]).

However, a distinction can be made between the "projection postulate" which corresponds merely to a *mathematical operation*[22] and the "quantum jump hypothesis"

[21] The quantum jump hypothesis is due to Dirac [34] and von Neumann [46].

[22] Notice that the standard projection postulate can be generalized so as to suit to the "positive operator-valued measures" (or POVM) formalism, which enables to describe all the possible measurements in

which is a possible *ontological interpretation* of this operation. While the quantum jump hypothesis has to be removed because of the problems it generates, the projection postulate can still be useful for the physicists practice.

More precisely, in most cases, the application of this mathematical operation to the vector formalism enables the physicists to take into account the empirical knowledge gained from a measurement and make predictions concerning further possible measurements. Concretely, it amounts to replace the state vector (or density operator) associated to the studied system immediately before the performed measurement by a new state vector which is equal to the eigenvector related to the value which has been obtained by means of this measurement. From this new state vector, the physicists can derive (by applying Born's rule) the occurrence probability of the possible outcomes of a second measurement. In other words, in most cases, the application of the projection postulate enables the physicists to fulfil the pragmatic function F_7.

This is true in *most* cases but not in *all* cases. Indeed, let us consider an ordinary measurement on a microscopic system S by means of a measurement apparatus M. From a strict theoretical point of view, in order to describe this measurement, we should also take into account the environment noted E, that is to say, we should apply Schrödinger's equation to the state vector (or density operator) associated to the whole system $S + M + E$. A consequence of the application of the projection postulate is that the terms of coherence between S and M are neglected. As it appears in the decoherence theory, these terms become extremely small in a very short time (see e.g. Zurek [56], Giulini [57], and Blanchard [58]). Nevertheless, they never become strictly equal to zero, so that *in principle* they still have testable consequences. This means that the application of the projection postulate does *not* enable the physicists to make predictions concerning measurements which would (if they were performed) point out the residual terms of coherence between S and M. Thus, the pragmatic function F_7 is not entirely fulfilled.

Let us remark that the current measurement apparatuses are largely dissipative systems. Accordingly, the residual terms of coherence between a microscopic system and a measurement apparatus are *in practice* unobservable (see e.g. [17, Chap. 7]). This means that the application of the projection postulate enables the physicists to make predictions concerning all further measurements *that nowadays can effectively be performed*, but not concerning all further measurements *that could be performed in principle*.

In order not to narrow the predictive power of the vector formalism, the projection postulate can be substituted by another mathematical operation, namely the *probability conditionalisation rule*. This rule makes it possible to determine, *conditional* on the occurrence of a certain outcome of a first measurement, the occurrence probability of the possible outcomes of a second measurement (for a precise mathematical formulation of this rule, see e.g. [13, p. 2292]). Since it doesn't modify the state vector (or density operator) associated to the whole system $S + M + E$, this rule does not neglect the above mentioned coherence terms. For this reason, only the probability conditionalisation rule makes it possible, *in all cases*, to fulfil the pragmatic function F_7.

microphysics, including "unsharp" or "approximate" measurements (see e.g. [30, p. 153, 31, p. 1008]). For simplicity, we will restrict the discussion to the standard (non-generalized) projection postulate.

Note however that this rule has a practical disadvantage: it compels the physicists to make calculations with the state vector (or density operator) associated to the system composed of all the microscopic systems and measurement apparatuses with which the system under study has interacted in the past. Such calculations can quickly become unfeasible. This is why many experimental physicists (by contrast with theoretical physicists which are more concerned with the predictive power of the state vector) still make use of the projection postulate (which remains relevant with regard to the current measurements).

5 Conclusion and Discussion

The main result of this investigation is the following: all the terms and mathematical components of QM can be interpreted merely in the light of the physicists' practice. This means, there is no need to connect these terms and mathematical components to elements (or features) assumed to belong (resp. to characterize) the world as it is in itself.

In particular, there is no need to interpret the state vector associated to a microscopic system as the representative of its physical state. By removing this interpretation we are no more compelled to invent mysterious and unobservable things such as the "superposition of physical states" or the "entanglement of physical states" assumed to occur in the world in itself.

As a consequence, the unsolvable measurement problem does not even arise. In other words, this problem can be considered as dissolved. Indeed, it is no more question of explaining the reduction of the physical state of a microscopic system during a measurement. Or to state it in the frame of the quantum description of the measurement, we are no more led to the paradoxical conclusion that the pointer of the measurement apparatus (or Schrödinger's cat) ends up in a superposition of physical states corresponding to macroscopically distinct positions (resp. in a superposition of being dead and being alive). In other words, there is no conflict with what the physicists in their labs are observing as a matter of fact, namely the pointer ending up in one specific position (resp. the cat as being either dead or alive). Note that this does not amount to say that the pointer "really" ends up in one specific position (resp. that the cat is "really" dead or alive). The pragmatist interpretation of QM does not tell us anything about such ontological statements.

What this paper is assumed to provide is an interpretation of QM which captures the actual physicists' practice, which therefore does not remove any term of this theory (e.g. it does not remove the expression "microscopic system" contrarily to instrumentalist interpretations of QM), and which is devoid of any conceptual or theoretical problem.

This interpretation does not deny the assumption that the world exists independently of the possible representations we can build of it, nor does it prevent physicists from making ontological conjectures concerning this world. After all, the search for a "faithful" representation of the world has always been in the history of science a source of motivation. Nevertheless, any metaphysical discussion concerning the world as it is in itself goes beyond the scope of a pragmatist interpretation of QM, which remains agnostic in this respect.

Let us remark finally that only a finite number of modifications of QM have been considered in this paper. Hence the program for a systematic pragmatist interpretation remains open. It consists in imagining all kinds of modifications of QM and checking if they are of some use or not for the physicists' practice.

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